Astro 501, Fundamental Astronomy — Problem Set 4

Due Oct 7

1. Stars spend more than 90% of the lives on the main sequence. So is it fair to say that stars burn most of their fuel on the main sequence? Let's consider this question for the Sun.

a) First, assume that the Sun will live $\tau_\odot = 10$ Gyr on the main sequence and during this time, its average luminosity will be $\mathcal{L} = 1\mathcal{L}_\odot$. How much energy will be produced during the main sequence phase, and what will be the mass of the helium core at the end of this phase?

The fusion of hydrogen to helium produces $Q = 6.3 \times 10^{18}$ ergs g^{-1}, or 1.25×10^{52} ergs $\mathcal{M}_\odot^{-1}$ of hydrogen. Since only 75% of the Sun's mass in hydrogen, this number becomes

$$Q = 9.4 \times 10^{51} \ \text{ergs}\ \mathcal{M}_\odot^{-1}$$

Meanwhile, over its main-sequence life, the Sun will emit

$$E_\odot = L_\odot \tau_\odot = 1.2 \times 10^{51} \text{ ergs}$$

Therefore, the Sun will fuse on the main sequence until its core reaches

$$\mathcal{M}_0 = E_\odot / QX \sim 0.13 \mathcal{M}_\odot$$

b) After turning off the main sequence, the Sun will become a shell-burning red giant star. This phase will last until the mass of the helium core reaches $\sim 0.45 M_\odot$. Estimate the amount of energy the Sun will produce as a red giant, and the lifetime of the phase. (Unlike main sequence stars, red giant branch objects have strong luminosity evolution during their lifetime. You will need to take this into account.)

Since the Sun is still burning hydrogen, the first part of this question is easy. The red giant phase stops at $\mathcal{M}_f = 0.45\mathcal{M}_\odot$, so

$$E_{RGB} = \left(\mathcal{M}_f - \mathcal{M}_0\right) \cdot Q \cdot X = 3 \times 10^{51} \text{ ergs}$$

or more than twice that on the main sequence. Also, during the RGB phase, the luminosity produced depends on the core mass, via

$$\mathcal{L} \propto \mathcal{M}_c^\gamma$$

with $\gamma \sim 8$. The core mass will therefore increase by

$$\frac{d\mathcal{M}_c}{dt} = \frac{\mathcal{L}}{XQ} = \frac{K\mathcal{M}_c^\gamma}{XQ}$$

The lifetime of the RGB phase is therefore given by

$$t = \int_{\mathcal{M}_0}^{\mathcal{M}_f} \frac{XQ}{K\mathcal{M}^\gamma} d\mathcal{M}$$

Now let $\mathcal{L}_0 = K M_0^\gamma$ be the luminosity of the star at the start of the giant branch phase. Then

$$t = \frac{XQ\mathcal{M}_0}{\mathcal{L}_0(1-\gamma)} \left\{ \left(\frac{\mathcal{M}_1}{\mathcal{M}_0} \right)^{1-\gamma} - 1 \right\} \sim 1.4 \times 10^9 \text{ yr}$$

c) Finally, assume that the Sun will end its life as a $0.57\mathcal{M}_\odot$ carbon white dwarf. What fraction of the Sun's energy will be released after the main sequence phase?

First, note that the energy produced by fusing helium to carbon is only $\sim 3\%$ that produced by fusing hydrogen to helium. So let's neglect the enegy produced by the helium and heavy elements in the initial composition of the star. If we do that, the computation of the total energy produced over the history of the star is straightforward.

The atomic weight of carbon is 12; the atomic weight of hydrogen is 1.0078. Therefore the energy released by fusing hydrogen to carbon is

$$\Delta E = \left(12 m_H - m_{C_{12}} \right) c^2 = 87.5 \text{ MeV}$$

per carbon atom. The total energy generated is therefore

$$E_{\text{tot}} = 0.57\mathcal{M}_\odot \cdot X\, N_{\text{Avo}}/12 = 6 \times 10^{51} \text{ ergs}$$

This means that only $\sim 20\%$ of the Sun's energy will be produced while it's on the main sequence.

2. White dwarfs created by first-generation stars (i.e., objects created shortly after the Big Bang) have had a long time to cool off and fade away. But how faint are they? To do the calculation, you'll need two factoids.

First, as first shown by Chandrasekhar (1939) in his book "Stellar Structure", the specific heat at constant volume of a fully degenerate object is given by

$$c_V = \frac{3}{2} \frac{N_{\text{Avo}} k}{\mu_I}$$

Second, the (*very* thin) atmosphere of the white dwarf is not degenerate, and acts much like any other atmosphere. In other words, in order for the core of the star to cool, its heat must get transported through the atmosphere, and, since the star is hot, this energy transport happens radiatively. With appropriate substitutions (and you are welcome to try it, if you like), the combination of the ideal gas law, Kramer's law opacity, and the equation for radiative energy transport (i.e., ∇_{rad}) yields a simple relation between the luminosity of a white dwarf and its (almost) isothermal core temperature

$$\mathcal{L} = C_1 \mathcal{M} T_c^{7/2}$$

where, if $\mathcal{L}$ and $\mathcal{M}$ are expressed in solar units, $C_1 \sim 5 \times 10^{-30}$.

Now consider a $5M_\odot$ star formed shortly after the Big Bang, 13 Gyr ago. From the initial-mass final-mass relation, such an object will form a $\sim 1M_\odot$ white dwarf.

a) Estimate the luminosity of the white dwarf today. Is the timescale for main sequence evolution important for this calculation?

First, let's quickly consider the lifetime of the star before it became a white dwarf. A lifetime of a main sequence star can be roughly estimated via

$$\tau \propto \frac{\mathcal{M}}{\mathcal{L}} \propto \frac{\mathcal{M}}{\mathcal{M}^\eta} \propto \mathcal{M}^{1-\eta}$$

where $\eta \sim 3.5$. From scaling to the Sun, the constant of proportionality is $\sim 10^{10}$ yr, so a $5M_\odot$ lives $\sim 1.8 \times 10^8$ yr (and the post-main sequence lifetime is much shorter than this). Since this is much, much shorter than the age of the universe, the pre-white dwarf lifetime of this star is negligible.

Now, let's examine the white dwarf. The energy contained in the white dwarf is

$$E = \mathcal{M}\, c_V T_c \Longrightarrow \mathcal{L} = -\frac{dE}{dt} = -\mathcal{M}\, c_V \frac{dT}{dt}$$

and the luminosity emitted is

$$\mathcal{L} = C_1 \mathcal{M}\, T_c^{7/2}$$

Now if you take the time derivative of this last equation

$$\frac{d\mathcal{L}}{dt} = \frac{7}{2} C_1 \mathcal{M}\, T_c^{5/2} \frac{dT_c}{dt} = \frac{7}{2} C_1 \mathcal{M} \left(\frac{\mathcal{L}}{C_1 \mathcal{M}}\right)^{5/7} \left(-\frac{\mathcal{L}}{\mathcal{M} c_V}\right)$$

or

$$\frac{d\mathcal{L}}{dt} = -\frac{7}{2c_V} C_1^{2/7} \mathcal{M}^{-5/7} \mathcal{L}^{12/7} \Longrightarrow \int \mathcal{L}^{-12/7} d\mathcal{L} = \int -\frac{7}{2c_V} C_1^{2/7} \mathcal{M}^{-5/7} dt$$

Integrating this over a Hubble time t_H yields

$$-\frac{7}{5} \mathcal{L}^{-5/7} \Big|_{\mathcal{L}_0}^{\mathcal{L}_1} = -\frac{7}{2c_V} C_1^{2/7} \mathcal{M}^{-5/7} t \Big|_0^{t_H}$$

so

$$\mathcal{L} = \left\{ \frac{5}{2} \frac{1}{c_V} C_1^{2/7} \mathcal{M}^{-5/7} t_H - \mathcal{L}_0^{-5/7} \right\}^{-7/5}$$

where $\mathcal{L}_0$ is the white dwarf luminosity when it starts cooling. Note, however, that $\mathcal{L}_0$ is relatively large number raised to a negative power. It is therefore negligible, and the above equation simplifies to

$$\mathcal{L} = \left(\frac{5t_H}{2c_V}\right)^{-7/5} \mathcal{M}\, C_1^{-2/5} = 2.1 \times 10^{-4} \mathcal{L}_\odot$$

b) The HETDEX project will be obtaining spectra for stars brighter than $m \sim 22$. How far away will HETDEX be able to identify these first-generation white dwarfs? (Note: the bolometric absolute magnitude of the Sun is $M = 4.74$.)

Absolute bolometric magnitude is related luminosity by

$$M = 4.74 - 2.5 \log \mathcal{L}$$

meaning that first generation white dwarfs have $M = 13.9$ Since HETDEX will observe down to $m = 22$, we have

$$(m - M) = 5 \log d - 5$$

giving a limiting distance of ~ 410 pc.